Warm-Up

$a=2 \quad b=-6 \quad c=1 \quad d=-2$

1) Find the multiplicative inverse of A. $A=\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

$\frac{1}{(2)(-2)-(-6)(1)}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$

$\frac{1}{-4+6}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$

$\begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}$

2) Find the product of AB when A= $\begin{bmatrix} -2 & 1 & -1 \\ -5 & 2 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ B= $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & 1 & 1 \end{bmatrix}$

$-2(1)+1(2)+-1(-1)$ $\quad$ $-2(0)+1(1)+-1(1)$ $\quad$ $-2(1)+1(3)+-1(1)$

$1 \quad 0 \quad 0$

$-5(1)+2(2)+-1(-1)$ $\quad$ $-5(0)+2(1)+-1(1)$ $\quad$ $-5(1)+2(3)+-1(1)$

$0 \quad 1 \quad 0$

$3(1)+-1(3)+1(1)$ $\quad$ $3(1)+-1(3)+1(1)$ $\quad$ $3(1)+-1(3)+1(1)$

$0 \quad 0 \quad 1$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Associated with every 2x2 square matrix is a real number, called its DETERMINANT.

Definition of the Determinant of a 2 × 2 Matrix

The determinant of the matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is denoted by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and is defined by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

We also say that the **value** of the **second-order determinant** $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is $a_1b_2 - a_2b_1$.

Study Tip

To evaluate a second-order determinant, find the difference of the product of the two diagonals.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

EXAMPLE 1 **Evaluating the Determinant of a 2 × 2 Matrix**

Evaluate the determinant of each of the following matrices:

a. $\begin{bmatrix} 5 & 6 \\ 7 & 3 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & 4 \\ -3 & -5 \end{bmatrix}$.

$5(3)-7(6)$

$15-42$

-27

$2(-5)-(-3)(4)$

$-10+12$

2

Solving a Linear System in Two Variables Using Determinants

Cramer's Rule

If

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2, \end{cases}$$

then

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

where

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

$$x = \frac{D_x}{D} \quad \text{and} \quad y = \frac{D_y}{D}$$

EXAMPLE 2 Using Cramer's Rule to Solve a Linear System

Use Cramer's rule to solve the system:

$$\begin{cases} 5x - 4y = 2 \\ 6x - 5y = 1. \end{cases}$$

replace x's

replace y's

$$D = \begin{vmatrix} 5 & -4 \\ 6 & -5 \end{vmatrix} \qquad D_x = \begin{vmatrix} 2 & -4 \\ 1 & -5 \end{vmatrix} \qquad D_y = \begin{vmatrix} 5 & 2 \\ 6 & 1 \end{vmatrix}$$

$5(-5) - 6(-4)$

$-25 + 24$

-1

$2(-5) - 1(-4)$

$-10 + 4$

-6

$5(1) - 6(2)$

$5 - 12$

-7

$$X = \frac{D_x}{D} = \frac{-6}{-1} = 6 \qquad Y = \frac{D_y}{D} = \frac{-7}{-1} = 7$$

$(6, 7)$

The Determinant of a 3 × 3 Matrix

Associated with every square matrix is a real number called its determinant. The determinant for a 3 × 3 matrix is defined as follows:

Definition of the Determinant of a 3 × 3 Matrix

A third-order determinant is defined by

Subtract.

Add.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

Each a on the right comes from the first column.

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$\begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ \cancel{a_2} & b_2 & c_2 \\ \cancel{a_3} & b_3 & c_3 \end{vmatrix} \quad \begin{vmatrix} \cancel{a_1} & b_1 & c_1 \\ \cancel{a_2} & \cancel{b_2} & \cancel{c_2} \\ \cancel{a_3} & b_3 & c_3 \end{vmatrix} \quad \begin{vmatrix} \cancel{a_1} & b_1 & c_1 \\ \cancel{a_2} & b_2 & c_2 \\ \cancel{a_3} & \cancel{b_3} & \cancel{c_3} \end{vmatrix}$$

EXAMPLE 3 **Evaluating the Determinant of a 3 × 3 Matrix**

Evaluate the determinant of the following matrix:

$$\begin{bmatrix} 4 & 1 & 0 \\ -9 & 3 & 4 \\ -3 & 8 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 4 & 1 & 0 \\ -9 & 3 & 4 \\ -3 & 8 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 & 0 \\ -9 & 3 & 4 \\ -3 & 8 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 & 0 \\ -9 & 3 & 4 \\ -3 & 8 & 1 \end{bmatrix}$$

$$4\begin{vmatrix} 3 & 4 \\ 8 & 1 \end{vmatrix}, \quad -9\begin{vmatrix} 1 & 0 \\ 8 & 1 \end{vmatrix}, \quad -3\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix}$$

$$\begin{vmatrix} 4 & 1 & 0 \\ -9 & 3 & 4 \\ -3 & 8 & 1 \end{vmatrix} = 4\begin{vmatrix} 3 & 4 \\ 8 & 1 \end{vmatrix} - (-9)\begin{vmatrix} 1 & 0 \\ 8 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix}$$

$$\begin{aligned} &= 4(3\cdot 1 - 8\cdot 4) + 9(1\cdot 1 - 8\cdot 0) - 3(1\cdot 4 - 3\cdot 0) \\ &= 4(3 - 32) + 9(1 - 0) - 3(4 - 0) \\ &= 4(-29) + 9(1) - 3(4) \\ &= -116 + 9 - 12 \\ &= -119 \end{aligned}$$

Solving Three Equations in Three Variables Using Determinants
Cramer's Rule

If $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$ then

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D},\ \text{and } z = \frac{D_z}{D},\ \text{where } D \neq 0.$$

These four third-order determinants are given by

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ These are the coefficients of the variables x, y, and z.

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$ Replace x-coefficients in D with the constants on the right of the three equations.

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$ Replace y-coefficients in D with the constants on the right of the three equations.

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$ Replace z-coefficients in D with the constants on the right of the three equations.

EXAMPLE 5 Using Cramer's Rule to Solve a Linear System in Three Variables

Use Cramer's rule to solve:

$$\begin{cases} x + 2y - z = -4 \\ x + 4y - 2z = -6 \\ 2x + 3y + z = 3. \end{cases}$$

$$D_x = -4\begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} - (-6)\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix}$$

$$-4(4+6) + 6(2+3) + 3(-4+4)$$

$$-4(10) + 6(5) \cancel{+3(0)}$$

$$-40 + 30 = -10 \qquad D_x = -10$$

$$D_y = 1\begin{vmatrix} -6 & -2 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} -4 & -1 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} -4 & -1 \\ -6 & -2 \end{vmatrix}$$

$$1(-6+6) - 1(-4+3) + 2(8-6)$$

$$1 + 4 = 5 \qquad D_y = 5$$

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 4 & -2 \\ 2 & 3 & 1 \end{vmatrix} \quad D_x = \begin{vmatrix} -4 & 2 & -1 \\ -6 & 4 & -2 \\ 3 & 3 & 1 \end{vmatrix} \quad D_y = \begin{vmatrix} 1 & -4 & -1 \\ 1 & -6 & -2 \\ 2 & 3 & 1 \end{vmatrix} \quad D_z = \begin{vmatrix} 1 & 2 & -4 \\ 1 & 4 & -6 \\ 2 & 3 & 3 \end{vmatrix}$$

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 4 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix}$$

$$= 1(4+6) - 1(2+3) + 2(-4+4)$$
$$= 1(10) - 1(5) + 2(0) = 5$$

$*D = 5$

Solve for D_x, D_y, and D_z

$\frac{D_x = -10}{5}$ $\quad \frac{D_y = 5}{5}$ $\quad \frac{D_z = 20}{5}$

x=-2, y=1, z=4

$$D_z = 1\begin{vmatrix} 4 & -6 \\ 3 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & -4 \\ 3 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & -4 \\ 4 & -6 \end{vmatrix}$$

$$1(12+18) - 1(6+12) + 2(\underbrace{-12+16}_{4})$$

$$30 - 18 + 8 = 20 \qquad D_z = 20$$