11.4 Introduction to Derivatives

Slope of the Tangent Line to a Curve at a Point

The **slope of the tangent line** to the graph of a function $y = f(x)$ at $(a, f(a))$ is given by

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that this limit exists. This limit also describes

- the **slope of the graph of f at $(a, f(a))$.**
- the **instantaneous rate of change of f with respect to x at a.**

EXAMPLE 1 **Finding the Slope of a Tangent Line**

a $f(a)$

Find the slope of the tangent line to the graph of $f(x) = x^2 + x$ at $(2, 6)$.

$$m_{\tan} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^2 + (2+h) - 6}{h}$$

$$\lim_{h \to 0} \frac{\cancel{4} + 4h + h^2 + \cancel{2} + h - \cancel{6}}{h} = \lim_{h \to 0} \frac{h^2 + 5h}{h} = \lim_{h \to 0} \frac{\cancel{h}(h+5)}{\cancel{h}}$$

$$\lim_{h \to 0} h + 5 = 0 + 5 = 5 \qquad m_{\tan} = 5$$

EXAMPLE 2 **Finding the Slope-Intercept Equation of a Tangent Line**

Find the slope-intercept equation of the tangent line to the graph of $f(x) = \sqrt{x}$ at $(4, 2)$.

$y = mx + b$

Foil

$$m_{\tan} = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} \left(\frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \right)$$

$$m_{\tan} = \lim_{h \to 0} \frac{\cancel{4} + h - \cancel{4}}{h(\sqrt{4+h} + 2)} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{4+h} + 2)} = \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2}$$

$$m_{\tan} = \frac{1}{\sqrt{4} + 2} = \frac{1}{2+2} = \frac{1}{4}$$

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{1}{4}(x - 4)$

$y - 2 = \frac{1}{4}x - 1$

$+2 \qquad +2$

$y = \frac{1}{4}x + 1$

Definition of the Derivative of a Function

Let $y = f(x)$ denote a function f. The **derivative of f at x**, denoted by $f'(x)$, read "f prime of x," is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

provided that this limit exists. The derivative of a function f gives the slope of f for any value of x in the domain of f'.

EXAMPLE 3 **Finding the Derivative of a Function**

a. Find the derivative of $f(x) = x^2 + 3x$ at x. That is, find $f'(x)$.

b. Find the slope of the tangent line to the graph of $f(x) = x^2 + 3x$ at $x = -2$ and at $x = -\frac{3}{2}$.

a.) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) - (x^2 + 3x)}{h}$$

$$\lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{+3x} + 3h \cancel{-x^2} \cancel{-3x}}{h} = \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(2x + h + 3)}{\cancel{h}} = \lim_{h \to 0} 2x + h + 3 = 2x + 3$$

$$f'(x) = 2x + 3$$

b.) $x = -2$

$f'(-2) = 2(-2) + 2$

$-4 + 2$

$f'(-2) = -2$

$x = -3/2$

$f'(-3/2) = \cancel{2}(-3/\cancel{2}) + 2$

$f'(-3/2) = -3 + 2$

$f'(-3/2) = -1$